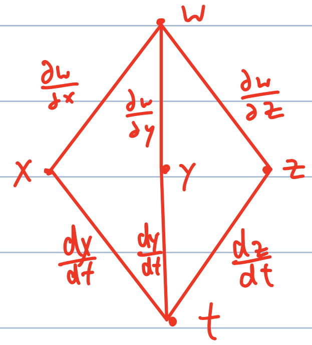
MATH2010E TUT08

Chain Rule: One Independent Variable

In Exercises 1–6, **(a)** express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then **(b)** evaluate dw/dt at the given value of t.



6. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}; \quad t = 1$

Ans: a) $\frac{\partial w}{\partial x} = -y\cos(xy)$, $\frac{\partial w}{\partial y} = -x\cos(xy)$, $\frac{\partial w}{\partial z} = 1$

$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{1}{t}$, $\frac{dz}{dt} = e^{t-1}$

By chain rule,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= -(\ln t)\cos(t\ln t)\cdot 1 - t\cos(t\ln t)\cdot\frac{1}{t} + 1\cdot e^{t-1}$$

$$= -(\ln t + 1)\cos(t\ln t) + e^{t-1}$$

OTOH, $w = e^{t-1} - \sin(t\ln t)$

$$\Rightarrow \frac{dw}{dt} = e^{t-1} - \cos(t\ln t)\left(\ln t + t\cdot\frac{1}{t}\right)$$

$$= e^{t-1} - \cos(t\ln t)(\ln t + 1)$$

b) $$\left.\frac{dw}{dt}\right|_{t=1} = -(0+1)\cos(1\cdot 0) + e^0$$

$$= -1 + 1$$

$$= 0$$

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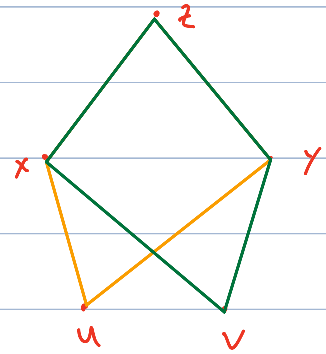
Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).



7. $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v;$ $(u, v) = (2, \pi/4)$

Ans: a) By chain rule,

$$\begin{cases} \dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u} \\ \dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v} \end{cases}$$

or

$$\begin{pmatrix} \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} 4e^x \ln y & 4e^x/y \end{pmatrix}\Bigg|_{\substack{x=\ln(u\cos v)\\ y=u\sin v}} \begin{pmatrix} \dfrac{1}{u} & -\tan v \\ \sin v & u\cos v \end{pmatrix}$$

$$= \begin{pmatrix} 4u\cos v \ln(u\sin v) & 4\cot v \end{pmatrix} \begin{pmatrix} \dfrac{1}{u} & -\tan v \\ \sin v & u\cos v \end{pmatrix}$$

OTOH,

$$z(u,v) = 4\exp(\ln(u\cos v))\ln(u\sin v)$$

$$= 4u\cos v \cdot \ln(u\sin v)$$

$$\Rightarrow \frac{\partial z}{\partial u} = 4\cos v \ln(u\sin v) + 4u\cos v\cdot\frac{\sin v}{u\sin v}$$

$$= 4\cos v\ln(u\sin v) + 4\cos v$$

$$\frac{\partial z}{\partial v} = -4u\sin v\cdot\ln(u\sin v) + 4u\cos v\cdot\frac{u\cos v}{u\sin v}$$

$$= -4u\sin v\ln(u\sin v) + 4u\cos v\cdot\cot v$$

b) At $(2, \pi/4)$,

$$\frac{\partial z}{\partial u} = \frac{4}{\sqrt{2}}\ln(\sqrt{2}) + \frac{4}{\sqrt{2}} = \sqrt{2}(\ln 2 + 2)$$

$$\frac{\partial z}{\partial v} = -4(\sqrt{2})\ln\sqrt{2} + 4(\sqrt{2})\cdot 1 = -2\sqrt{2}\ln 2 + 4\sqrt{2}$$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x, y, and z both by using the Chain Rule and by expressing u directly in terms of x, y, and z before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z).

11. $u = \dfrac{p-q}{q-r}$, $\quad p = x + y + z$, $\quad q = x - y + z$,
$r = x + y - z$; $\quad (x, y, z) = \left(\sqrt{3}, 2, 1\right)$

Ans: a) $\left(\frac{\partial u}{\partial p} \quad \frac{\partial u}{\partial q} \quad \frac{\partial u}{\partial r}\right) = \left(\frac{1}{q-r} \quad \frac{r-p}{(q-r)^2} \quad \frac{p-q}{(q-r)^2}\right)$

$$\begin{pmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

By chain rule,

$$\left(\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial z}\right) = \frac{1}{(q-r)^2}\left(q-r \quad r-p \quad p-q\right)\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{(q-r)^2}\left(0 \quad 2p-2r \quad 2q-2p\right)$$

$$= \frac{1}{(2z-2y)^2}\left(0 \quad 4z \quad -4y\right)$$

$$= \frac{1}{(z-y)^2}\left(0 \quad z \quad -y\right)$$

OTOH, $u = \dfrac{p-q}{q-r} = \dfrac{2y}{2z-2y} = \dfrac{y}{z-y}$

$\Rightarrow \quad \dfrac{\partial u}{\partial x} = 0$

$\dfrac{\partial u}{\partial y} = \dfrac{1}{(z-y)^2}\Big((z-y) - y(-1)\Big) = \dfrac{z}{(z-y)^2}$

$\dfrac{\partial u}{\partial z} = -\dfrac{y}{(z-y)^2}$

b) At $(x, y, z) = (\sqrt{3}, 2, 1)$,

$\dfrac{\partial u}{\partial x} = 0$, $\quad \dfrac{\partial u}{\partial y} = 1$, $\quad \dfrac{\partial u}{\partial z} = -2$ //

40. Assume that $w = f\left(ts^2, \frac{s}{t}\right)$, $\frac{\partial f}{\partial x}(x, y) = xy$, and $\frac{\partial f}{\partial y}(x, y) = \frac{x^2}{2}$.

Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$.

Ans: Let $x = ts^2$, $y = \frac{s}{t}$

Then $\frac{\partial x}{\partial t} = s^2$, $\frac{\partial x}{\partial s} = 2ts$, $\frac{\partial y}{\partial t} = -\frac{s}{t^2}$ $\frac{\partial y}{\partial s} = \frac{1}{t}$

By chain rule,

$$\begin{pmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{pmatrix}$$

$$= \begin{pmatrix} ts^2 \cdot \frac{s}{t} & \frac{1}{2}(ts)^2 \end{pmatrix} \begin{pmatrix} s^2 & 2ts \\ -\frac{s}{t^2} & \frac{1}{t} \end{pmatrix}$$

$$= \begin{pmatrix} s^5 - \frac{1}{2}s^5 & 2ts^4 + \frac{1}{2}ts^4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}s^5 & \frac{5}{2}ts^4 \end{pmatrix}$$

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44. Polar coordinates Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $w = f(x, y)$.

a. Show that

$$\frac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta.$$

b. Solve the equations in part (a) to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial\theta$.

c. Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

Ans: a) By chain rule,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta)$$

b)
$$\begin{pmatrix} \frac{\partial w}{\partial r} \\ \frac{1}{r}\frac{\partial w}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

So
$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial w}{\partial r} \\ \frac{1}{r}\frac{\partial w}{\partial \theta} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial r} \\ \frac{1}{r}\frac{\partial w}{\partial \theta} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial w}{\partial \theta}\sin\theta \\ \frac{\partial w}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial w}{\partial \theta}\cos\theta \end{pmatrix}$$

c)
$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial w}{\partial \theta}\sin\theta\right)^2 + \left(\frac{\partial w}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial w}{\partial \theta}\cos\theta\right)^2$$

$$= \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 + 0$$

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Differentiating Integrals Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x)\, dt,$$

(*) then $F'(x) = \int_a^b g_x(t, x)\, dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t, x)\, dt$$

by letting

$$G(u, x) = \int_a^u g(t, x)\, dt, \quad \Longrightarrow \quad \begin{cases} \dfrac{\partial G}{\partial u} = g(u,x) & \text{by FTC} \\ \dfrac{\partial G}{\partial x} = \displaystyle\int_a^u g_x(t,x)\,dt & \text{by (*)} \end{cases}$$

where $u = f(x)$. Find the derivatives of the functions in Exercises 51 and 52.

51. $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}\, dt$ **52.** $F(x) = \int_{x^2}^{1} \sqrt{t^3 + x^2}\, dt$

Ans: By chain rule,

$$\frac{dG}{dx} = \frac{\partial G}{\partial u}\cdot\frac{du}{dx} + \frac{\partial G}{\partial x}\cdot\cancel{\frac{dx}{dx}}\ 1$$

$$= g(u,x)\cdot\frac{du}{dx} + \int_a^u g_x(t,x)\,dt$$

51) $g(t,x) = \sqrt{t^4+x^3}$, $u = f(x) = x^2$

So $F'(x) = \dfrac{d}{dx} G(f(x), x) = g(f(x),x)\cdot f'(x) + \displaystyle\int_0^{f(x)} g_x(t,x)\,dt$

$$= \sqrt{(x^2)^4 + x^3}\cdot(2x) + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4+x^3}}\,dt$$

$$= 2x\sqrt{x^8+x^3} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4+x^3}}\,dt \quad //$$